Journal of Statistical Physics, Vol. 63, Nos. 1/2, 1991

Metastable Systems Driven by Colored Noise: The Stationary State

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Received May 21, 1990; final October 16, 1990

A new perturbation method of finding the stationary distribution function of the form $P = R\exp(-\phi/D)$ for a metastable (anharmonic) system driven by exponentially correlated noise is presented. The noise term is modeled by a Langevin equation and the stationary solution of the resultant $(2+1)$-dimensional Fokker–Planck equation is sought as a series expansion in the anharmonicity parameter around the known harmonic solution valid near the metastable minimum. The series converges for small τ in the leading order of the noise intensity D anywhere within the well. Analytic expressions for ϕ and R were found for a metastable and a bistable potential. The resultant decay rate is in accordance with previously published results. The method is suitable also for numerical calculations.

KEY WORDS: Stationary state; Gaussian colored noise; first passage time.

The problem of the stochastic dynamics of a metastable $(1+1)$-dimensional system driven by exponentially correlated Gaussian noise (correlation time τ) has attracted a good deal of attention in recent years, in particular with regard to the decay rate κ_τ. The consensus[1] emerging from the many conflicting theories is that the Arrhenius factor of the decay rate increases as $O(\tau^2/D)$ for small τ and noise intensity D. This result was derived by instanton methods,[2,3] by considering the Stratonovich type[4] Fokker–Planck equation,[1] and by embedding the $(1+1)$-dimensional system driven by colored noise in a $(2+1)$-dimensional one driven by white noise,[5] an approach adopted also in this work. A common feature of these small-τ calculations is an expansion around the Markovian stationary state. The present work approaches the problem from the opposite side. It is assumed that the anharmonic potential is given by a

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0022-4715/91/0400-0389$06.50/0

polynomial and the full stationary solution is then sought in the form of a perturbation series in the anharmonicity parameter around the known (for any τ) harmonic stationary state valid near the well minimum. The coefficients of the expansion are sought as analytic functions of τ; the proposed method is, however, also well suited for numerical calculations.

It turns out that for small enough correlation time τ and noise intensity D the perturbation series converges for arbitrary values of the anharmonicity parameter and yields the stationary distribution everywhere within the well in the leading order of D. The actual domain of convergence in the (D, τ) plane could not be determined; the validity of the expansion was verified *a posteriori* by comparison with published results.

A stationary distribution function in the extended two-dimensional phase space was derived in this fashion for a metastable (10) and a bistable (13) potential; the resultant barrier height Q_τ is in accordance with the general formula given by Bray *et al.*[3] Once the stationary distribution is known, it is easy to derive, *inter alia*, the mean first passage time and the corresponding decay rate.

Consider the stochastic equation

$$\dot{x} = -V'(x) + \xi(t) \tag{1}$$

where ξ is an exponentially correlated Gaussian noise such that $\langle \xi(t_1)\, \xi(t_2) \rangle = D\tau^{-1} \exp(-|t_1 - t_2|/\tau)$. It is possible to embed this $(1+1)$-dimensional system driven by colored noise into a $(2+1)$-dimensional one driven by white noise.[5] This is achieved by supplementing Eq. (1) with

$$\dot{\xi} = -\xi/\tau + (2D\tau^{-2})^{1/2}\, \dot{w}(t) \tag{2}$$

where $\dot{w}(t)$ is the standard Wiener process. The (Ito) Fokker–Planck equation for the extended system (1) and (2) is then

$$\frac{\partial P}{\partial t} = L_\tau[P] = -\frac{\partial}{\partial x}\{[-V'(x) + \xi]\, P\} + \frac{\partial}{\partial \xi}\left[\frac{\xi}{\tau} P\right] + \frac{D}{\tau^2}\frac{\partial^2 P}{\partial \xi^2} \tag{3}$$

As $\tau \to 0$, Eq. (1) leads to a Smoluchowski equation whose stationary state is given by the thermal equilibrium distribution $P_T = \exp[-V(x)/D]$. For $\tau > 0$ the condition of detailed balance is no longer satisfied and P_T is not a stationary state. The problem is then to find a solution of $L_\tau[P(x, \xi \mid \tau)] = 0$ such that $P(x, \xi \mid 0) \equiv P_T(x)$ for all ξ.

A simple example is furnished by the harmonic potential $V(x) = \omega x^2/2$ $(\omega > 0)$, for which the Eqs. (1) and (2) can be solved explicitly. The quantities relevant to the stationary state are the equal-time correlations as $t \to \infty$: $\langle \xi^2(t) \rangle \to D/\tau$, $\langle x(t)\, \xi(t) \rangle \to D/(1 + \omega\tau)$, and $\langle x^2(t) \rangle \to$

$D/[\omega(1+\omega\tau)]$. The corresponding unnormalized solution of the equation $L_\tau[P]=0$ is then the bivariate Gaussian probability distribution[6] $P_0(x, \xi \mid \tau) = \exp[-\phi^{(0)}(x, \xi \mid \tau)/D]$ with

$$\phi^{(0)}(x, \xi \mid \tau) = \frac{\tau}{2}(1+\omega\tau)\,\xi^2 - \omega\tau(1+\omega\tau)\,\xi x + \frac{\omega}{2}(1+\omega\tau)^2 x^2 \tag{4}$$

Locally P_0 is a solution of $L_\tau[P_0]=0$ for arbitrary ω.

Let now the stationary solution of Eq. (3) be sought in the form[5]

$$P(x, \xi \mid \tau) = R(x, \xi \mid \tau)\exp[-\phi(x, \xi \mid \tau)/D]$$

where the stochastic potential ϕ and the multiplicative factor R satisfy

$$\frac{1}{D} R\mathscr{L}_\tau^{(-1)}[\phi] + \mathscr{L}_\tau^{(0)}[R, \phi] + DR_{\xi\xi} = 0 \tag{5}$$

where the operators $\mathscr{L}_\tau^{(i)}$ are

$$\mathscr{L}_\tau^{(-1)}[\phi] = \phi_\xi^2 + \xi\tau(\phi_x\tau - \phi_\xi) - V'\tau^2\phi_x \tag{6}$$

$$\mathscr{L}_\tau^{(0)}[R, \phi] = \tau^2(V' - \xi)\,R_x + (\xi\tau - 2\phi_x)\,R_\xi + (V''\tau^2 + \tau - \phi_{\xi\xi})\,R \tag{7}$$

Let the stochastic potential be designated as ψ in the leading order of D. Then from Eq. (5) it follows that $\mathscr{L}_\tau^{(-1)}[\psi]=0$. The solution of this equation may then be used to obtain R in the leading order of the noise intensity from $\mathscr{L}_\tau^{(0)}[R, \psi]=0$. The basic idea of this article is to seek the stationary solution in the form of a perturbation series in the anharmonicity parameter around the harmonic solution (4) valid near the well minimum. The stochastic potential ψ is written accordingly as

$$\psi(x, \xi \mid \tau) = \sum_{i=0}^{\infty} b^i \psi^{(i)}(x, \xi \mid \tau) \tag{8}$$

and we demand that $\mathscr{L}_\tau^{(-1)}[\psi]=0$ be satisfied in each order of b separately. $\psi^{(0)} \equiv \phi^{(0)}$ is known and for $i>0$ the problem becomes linear. If ψ is found in this manner to a given order in b, then R may be found to the same order from $\mathscr{L}_\tau^{(0)}[R, \psi]=0$ by the same procedure using the ansatz

$$R(x, \xi \mid \tau) = 1 + \sum_{i=1}^{\infty} b^i R^{(i)}(x, \xi \mid \tau) \tag{9}$$

The functions ψ and R must go over to the Smoluchowski limit as $\tau \to 0$, i.e., $\psi(x, \xi \mid 0) \equiv V(x)$ and $R(x, \xi \mid 0) \equiv 1$.

It will be assumed throughout that $V(x)$ is a polynomial in x with a metastable or a bistable minimum at $x_0 = 0$ and a local maximum at the point x_1. It will be convenient to introduce the frequencies $\omega_i = V''(x_i)$ (note that $\omega_1 < 0$), the dimensionless times $\tau_i = |\omega_i|\,\tau$, and also a new variable $y = \tau\xi$ which has the dimension of x.

The simplest possible example, on which the basic features of the calculation will be outlined, is provided by the cubic potential

$$V(x) = ax^2/2 - bx^3/3 \tag{10}$$

with $\omega_0 = a$, $x_1 = a/b$, $\omega_1 = -a$, and $\tau_1 = a\tau$. The barrier height is $Q_0 = a^3/6b^2$ and $\psi^{(0)}$ is obtained from Eq. (4) with $\omega = \omega_0$.

Inspection reveals that $\psi^{(i)}(x, \xi\,|\,\tau) = (-1)^i\,\psi^{(i)}(-x, -\xi\,|\,\tau)$. This suggests that $\psi^{(i)}$ be sought in the form of either even- or odd-order homogeneous polynomials in x and ξ of order $j \leqslant i+2$. This ansatz follows from the form of $\psi^{(0)}$ and from the Smoluchowski limit. Their coefficients may then be found successively by substitution into $\mathscr{L}_\tau^{(-1)}[\psi] = 0$ and by comparison of factors at $x^n y^{n'}$ in each order of b. It turns out that the functions $\psi^{(i)}$ are given by homogeneous polynomials of the order $i+2$, a circumstance which greatly simplifies the calculation. In fact, there exists a direct method of evaluating the functions $\psi^{(i)}$, which obviates the need to solve a system of linear equations. It is presented in the Appendix; the results are summarized here. The small-τ analysis is more conveniently carried out in the (x, y) variables. Let $K_m(x, y\,|\,\tau)$ be a homogeneous polynomial of order m in the variables x and y: $K_m(x, y\,|\,\tau) = \sum_{j=0}^{m} k_j^{(m)}(\tau)\, x^j y^{m-j}$, such that the coefficients $k_j^{(m)}$ are rational functions of τ regular for $\tau \geqslant 0$ and nonvanishing at $\tau = 0$. With increasing perturbation order, $k_j^{(m)}$ become rapidly extremely complicated and cannot be presented here *in extenso*. The overall structure of the solutions is, however, quite simple. One has

$$\psi^{(i)}(x, y\,|\,\tau) = \tau^{i-1} K_{i+2}(x, y\,|\,\tau) \tag{11}$$

Moreover, here $\psi^{(1)}(x, \xi\,|\,0) = -x^3/3$ as expected. This scheme clarifies the behavior of $\psi(x, y\,|\,\tau)$ for short correlation times. For small enough τ the series (8) converges in the well within the radius $R \leqslant a/b$ for any b. In calculations it is therefore convenient to expand the functions $k_j^{(m)}(\tau)$ into a power series in τ and to retain only the desired number of terms.

In principle it is also possible to consider an expansion around the potential maximum, i.e., around the saddle point of ψ in the (x, ξ) plane. In this case, however, the coefficients $k_j^{(m)}(\tau)$ are singular on discrete sets S_m. Moreover, $S_m \subset S_{m+1}$ and the expansion diverges for all $\tau > 0$. In particular, the expressions (12) and (14) for the barrier height Q_τ cannot be obtained in this way. This circumstance makes the sadle point study computationaly very demanding, since beyond the quadratic approximation it can only be approached from the minimum.

For large τ the resultant functions $\psi^{(i)}$ are more easily described in the (x, ξ) plane: $\psi^{(i)}(x, \xi \mid \tau) = \tau^2 K_{i+2}(x, \xi \mid \tau^{-1})$. The domain of convergence in the (x, ξ) plane is in this case unclear. The τ^2 factor is at variance with the published large-τ behavior ($\sim \tau/D$) of the Arrhenius factor[2,7,8] and so only the small-τ behavior shall be considered in the sequel.

Small-τ analysis of the potential (10) shows that the function ψ always has a minimum at (0, 0) and a saddle at $(a/b, 0)$. The quantities of interest are the obvious equality $\psi(0, 0 \mid \tau) = 0$ and

$$\psi(x_1, 0 \mid \tau) = Q_0[1 + \tfrac{1}{5}\tau_1^2 - \tfrac{6}{35}\tau_1^4 + \tfrac{8}{21}\tau_1^6 - \tfrac{72}{55}\tau_1^8 + \cdots] \tag{12}$$

This is the barrier height Q_τ; it is in accordance with Eq. (15) of Ref. 3. The second derivatives of ψ at both extremum points are given for any (small) τ by the quadratic form $\psi^{(0)}$ only, i.e., by Eq. (4) with $\omega = \omega_i$ at $x = x_i$; all higher order terms in τ cancel identically. Comparison with Eq. (6) shows that this result holds for arbitrary τ, so that the expansion around the minimum yields the correct values near the saddle. One may thus conclude that the perturbations converge to the exact solution $\psi(x, \xi \mid \tau)$ within the radius $R \leqslant a/b$ for sufficiently small τ.

The archetypal bistable potential $V(x) = b^2x^4/4 - a^2x^2/2$ is amenable to the same procedure after a shift of the axes which brings one of the minima to the origin:

$$V(x) = a^2x^2 - abx^3 + b^2x^4/4 \tag{13}$$

Then $\omega_0 = 2a^2$ at the left minimum, $x_1 = a/b$, $\omega_1 = -a^2$, $\tau_1 = a^2\tau$, and the barrier height is $Q_0 = a^4/4b^2$. The solution $\psi(x, \xi \mid \tau)$ is again sought in the form (8) with $\psi^{(0)}$ given by Eq. (4) and $\omega = \omega_0$. The overall structure of ψ is similar to the preceding case, see Eq. (11); one gains, however, but one degree in τ for every two perturbation orders in b.

Small-τ analysis of the function $\psi(x, \xi \mid \tau)$ shows that it has extrema at $(x_i, 0)$ and that its second derivatives there are for all τ given by the quadratic term $\psi^{(0)}$ only, i.e., by Eq. (4) with $\omega = \omega_i$ at $x = x_i$. There is obviously $\psi(0, 0 \mid \tau) = 0$ and

$$\psi(x_1, 0 \mid \tau) = Q_0[1 + \tfrac{1}{2}\tau_1^2 - \tfrac{6}{5}\tau_1^4 + \tfrac{279}{35}\tau_1^6 - \cdots] \tag{14}$$

All third derivatives of ψ vanish identically at the saddle. This calculation to the sixth order in τ corresponds to the 14th order of the perturbation theory; four more orders are required to obtain the next nonvanishing (the eighth) power of τ in Q_τ. The function $\psi(x, 0 \mid \tau)$ to the tenth order in b (fourth order in τ) is listed explicitly in the Appendix. The expression (14) for Q_τ was derived previously by Bray *et al.*[3] using instanton methods.

The multiplicative factor R is obtained by the same perturbation procedure. For small τ, for the metastable potential (10) one has $R^{(i)}(x, y \mid \tau) = \tau^i K_i(x, y \mid \tau)$, whereas for the bistable potential (13) one gains, as for ψ, only one degree of τ for each two perturbation orders in b. For large τ one has $R^{(i)}(x, \xi \mid \tau) = K_i(x, \xi \mid \tau^{-1})$ and the domain of convergence in the (x, ξ) plane is again unclear. The quantity of interest for the intended decay rate calculation is the value $R(x_1, 0 \mid \tau)$. For the metastable potential (10) one obtains

$$R(x_1, 0 \mid \tau) = 1 - 3a\tau + 9a^2\tau^2/2 + O(\tau^3) \tag{15}$$

and for the bistable potential (13) this becomes

$$R(x_1, 0 \mid \tau) = 1 - 9a^2\tau^2/2 + O(\tau^2) \tag{16}$$

The corresponding function $R(x, 0 \mid \tau)$ is quoted in the Appendix. It should be noted that R does not have extrema at the points $(x_i, 0)$, but one has $R_{\xi\xi}(x_i, 0 \mid \tau) = 0$ at least for small τ. The function ψ has a saddle point at $(x_1, 0)$ for $|V''(x_1)| \tau < 1$, but the stationary distribution function $P = R \exp(-\psi/D)$ does not, for any $\tau > 0$, by virtue of this property of the function R.

With the stationary probability distribution $P(x, \xi \mid \tau)$ known it is possible to find an asymptotic (in small D) expression for the mean first passage time[5,9] $T(B)$ out of the domain of attraction B of the metastable well and also the corresponding decay rate. The calculation is standard and follows closely the work of Matkowsky *et al.*[9] so that only the main points need be presented here. $T(B)$ is sought asymptotically for small D and the exponent ψ is sufficient for this task. Its value is approximately known at the saddle, its second derivatives at $(x_i, 0)$ are known exactly. In the calculation it is convenient to introduce a scaled phase space (x, z) by the transformation $\xi = \omega_0 z$. The separatrix in the saddle point vicinity is then given by $z_\Gamma = -(x - x_1) \tan \varepsilon$, where $\tan \varepsilon = (1 + \tau_1)/\tau_1$, $\lim_{\tau \to 0} \varepsilon = \pi/2$ (Smoluchowski $1 + 1$ limit) and $\lim_{\tau \to \infty} \varepsilon = \pi/4$ (large-τ limit). Near the saddle along Γ the function ψ is approximately

$$\psi \approx Q_\tau + \frac{1}{2} \psi_{ss}(0, 0)\, s^2, \qquad \psi_{ss}(0, 0) = \frac{\omega_1^2 \tau (1 - \tau_1)^2}{\tau_1^2 + (1 + \tau_1)^2} \tag{17}$$

where s is the distance from the saddle measured along Γ. The decay rate is then

$$\kappa_\tau = \frac{1}{2T(B)} = \frac{(\omega_0 |\omega_1|)^{1/2}}{2\pi} \mathscr{B} e^{-Q_\tau/D} \tag{18}$$

$$\mathscr{B} = \frac{1+\tau_0}{1-\tau_1} R(x_1, 0 \mid \tau) = \frac{1+V''(x_0)\,\tau}{1+V''(x_1)\,\tau} R(x_1, 0 \mid \tau) \tag{19}$$

Q_τ is given for small τ by Eqs. (12) and (14) and $R(x_1, 0 \mid \tau)$ by Eqs. (15) and (16) for the two cases studied here. κ_0 reproduces correctly the Kramers result[9] for overdamped systems.

For the bistable potential (13) the numerical factor $\mathscr{B}$ leads to a result often quoted in the literature,[5,10]

$$\mathscr{B} = \frac{1+2a^2\tau}{1-a^2\tau}\left[1 - \frac{9}{2}a^2\tau + O(\tau^2)\right] \approx 1 - \frac{3}{2}a^2\tau$$

The crucial step of the calculation is the ansatz $P = R\exp(-\phi/D)$, which guarantees the convergence in the leading order of D, at least for small τ. The simpler ansatz $P = \exp(-\bar{\phi}/D)$ yields, of course, the same ψ. The next order of $\bar{\phi}$ in D, which corresponds to the factor R of the present ansatz, yields rather contradictory results in the saddle point vicinity, presumably due to a breakdown of the polynomial approximation. Either ansatz, however, confirms that the full stationary distribution function P *does not* have extrema at the points $(x_i, 0)$ of the (x, ξ) plane for any $\tau > 0$. The shift in their position is $O(D\tau)$, it does not alter the asymptotic expression for κ_τ. An extension of the proposed method to larger τ or D is contingent on a suitable ansatz for P for which the polynomial approximation converges. The study of the extremal properties of P in this limit merits further attention.

APPENDIX

In principle it is possible to find the coefficients of the homogeneous polynomials which are the solutions to Eq. (6) by direct substitution and successive solving of the resultant linear system in every order of b. A scaling argument allows one to calculate the coefficients directly. It will be illustrated on the bistable potential (13). Let $\rho = ax$, $\sigma = ay = a\tau\xi$, $\tau_1 = a^2\tau$, $\beta = b/a^2$, and $\psi = \sum_{i=0}^{\infty} \beta^i \psi^{(i)}(\rho, \sigma \mid \tau_1)$. Then in the kth order $\mathscr{L}_\tau^{(-1)}[\psi] = 0$ becomes

$$\begin{aligned} &\psi_\rho^{(k)}(\sigma - 2\tau_1\rho) - (\sigma - 2\tau_1\psi_\sigma^{(0)})\,\psi_\sigma^{(k)} \\ &\quad = -3\tau_1\rho^2\psi_\rho^{(k-1)} + \tau_1\rho^3\psi_\rho^{(k-2)} - \tau_1 \sum_{j=1}^{k-1} \psi_\sigma^{(j)}\psi_\sigma^{(k-j)} \end{aligned} \tag{20}$$

$\psi^{(0)}$ is known and suggests the introduction of new variables $u = 2\tau_1\rho - \sigma$, $v = 4\tau_1\rho - \sigma$. The term $\psi^{(k)}$ is a homogeneous polynomial of order $k+2$ in u and v and thus may be written as $\psi^{(k)} = v^{k+2}P_k(z)$, $z = u/v$, and $P_k(z)$ is a polynomial of order $k+2$ in z. In these variables Eq. (20) is

$$\begin{aligned}(k+2)\,P_k(z) + [1 - z(1-2\tau_1)]\,P_k'(z) \\ = -\frac{3}{2}(1-z)^2 X_{k-1} + \frac{1}{4\tau_1}(1-z)^3 X_{k-2} - \tau_1 \sum_{j=1}^{k-1} Y_j Y_{k-j}\end{aligned} \tag{21}$$

where $X_k = 2(k+2)\,P_k + (1-2z)\,P_k'$ and $Y_k = (k+2)\,P_k + (1-z)\,P_k'$. The particular solution of (21) is then easily found by direct integration. In fact, for $\tau_1 < 1/2$ one gets

$$P_k(\omega) = \gamma \sum_{i=0}^{k+2} \frac{a_i}{(k+2)\,\gamma - i}\, w^i \tag{22}$$

where $w = \gamma - z$, $\gamma = 1/(1-2\tau_1)$, and a_i are the coefficients of the polynomial expansion of the right-hand side of (21) in powers of w. Similarly for $\tau_1 \geqslant 1/2$. In fact, the singular character of (21) at $\tau_1 = 1/2$ suggests, even though this would be difficult to prove, the limit of convergence of the perturbations. These equations yield an efficient algorithm for artificial intelligence software. Similar equations may be derived for the function R. In principle the theory is applicable also to more then one expansion parameter.

For reference, I list the function $\psi(x, 0\,|\,\tau)$ to the fourth order in τ, i.e., to the tenth order in β, for $|x| \leqslant a/b$:

$$\psi(x, 0\,|\,\tau_1) = 4Q_0 \sum_{i=0}^{10} p_i(\beta\rho)^i$$

Here $\beta\rho = bx/a$ and the coefficients p_i are given by

$$p_0 = (2\tau_1 + 1)^2, \qquad p_1 = 48\tau_1^4 - 16\tau_1^3 - 20\tau_1^2 - 12\tau_1 - 1$$

$$p_2 = -(6080\tau_1^4 - 1056\tau_1^3 - 516\tau_1^2 - 156\tau_1 - 3)/12$$

$$p_3 = 2\tau_1(5026\tau_1^3 - 510\tau_1^2 - 120\tau_1 - 15)/5$$

$$p_4 = -\tau_1(12886\tau_1^3 - 774\tau_1^2 - 87\tau_1 - 3)/3$$

$$p_5 = 3\tau_1^2(1868\tau_1^2 - 64\tau_1 - 3), \qquad p_6 = -\tau_1^2(37508\tau_1^2 - 672\tau_1 - 9)/8$$

$$p_7 = 4\tau_1^3(633\tau_1 - 5), \qquad p_8 = -2\tau_1^3(6409\tau_1 - 15)/15$$

$$p_9 = 164\tau_1^4, \qquad p_{10} = -41\tau_1^4/3$$

This function yields the barrier height (12) to the fourth order in τ and also the values of $\psi_{xx}(x_i, 0 \mid \tau)$. The function $\psi^{(10)}(x, \xi \mid \tau)$ has $3+4+\cdots+13=88$ terms $x^j \xi^{i-j}$, $2 \leqslant i \leqslant 10$ and $0 \leqslant j \leqslant i$. If only the $O(\tau^2)$ terms are required, this reduces to 42 such terms in $\psi^{(6)}$ and only to 18 in $\psi^{(3)}$ for the metastable potential (10). In a plot of the results one observes a moderate increase in the barrier height accompanied by a fairly pronounced increase in the barrier width. The corresponding function $R(x, 0 \mid \tau)$ in the new variables is

$$R(x, 0 \mid \tau) = 1 - 9\tau_1 \beta\rho(1 - \beta\rho/2) + O(\tau_1^2)$$

ACKNOWLEDGMENTS

It is my pleasant duty to thank Prof. L. Gunther for his critical comments and encouragement and for his help with the preparation of this manuscript.

REFERENCES

1. P. Jung, P. Hänggi, and F. Marchesoni, *Phys. Rev. A* **40**:5447 (1989).
2. A. J. Bray and A. J. McKane, *Phys. Rev. Lett.* **62**:493 (1989).
3. A. J. Bray, A. J. McKane, and T. J. Newman, *Phys. Rev. A* **41**:657 (1990).
4. R. F. Fox, *J. Stat. Phys.* **46**:1145 (1987).
5. M. M. Klosek-Dygas, B. J. Matkowsky, and Z. Schuss, *SIAM J. Appl. Math.* **48**:425 (1988).
6. S. Chandrasekhar, *Rev. Mod. Phys.* **15**:1 (1943).
7. P. Jung and P. Hänggi, *Phys. Rev. A* **35**:4464 (1987).
9. B. J. Matkowsky, Z. Schuss, and E. Ben-Jacob, *SIAM J. Appl. Math.* **42**:835 (1982).
10. P. Hänggi, F. Marchesoni, and P. Grigolini, *Z. Phys. B-Condensed Matter* **56**:333 (1984).